The period-index problem for twisted topological K-theory*

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Abstract

We introduce and solve a period-index problem for the Brauer group of a topological space. The period-index problem is to relate the order of a class in the Brauer group to the degrees of Azumaya algebras representing it. For any space of dimension d, we give upper bounds on the index depending only on d and the order of the class. By the Oka principle, this also solves the period-index problem for the analytic Brauer group of any Stein space that has the homotopy type of a finite CW-complex. Our methods use twisted topological K-theory, which was first introduced by Donovan and Karoubi. We also study the cohomology of the projective unitary groups to give cohomological obstructions to a class being represented by an Azumaya algebra of degree n. Applying this to the finite skeleta of the Eilenberg-MacLane space $K(\mathbb{Z}/\ell, 2)$, where ℓ is a prime, we construct a sequence of spaces with an order ℓ class in Br, but whose indices tend to infinity.

Key Words Brauer groups, twisted K-theory, twisted sheaves, stable homotopy theory, cohomology of projective unitary groups.

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Contents

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1 Introduction

This paper gives a solution to a period-index problem for twisted topological K-theory. The solution should be viewed as an existence theorem for twisted vector bundles.

Let X be a connected CW-complex. An Azumaya algebra $\mathcal{A}$ on X is a non-commutative algebra over the sheaf $\mathcal{C}$ of complex-values functions on X such that $\mathcal{A}$ is a vector bundle of rank n^2, and the stalks are finite dimensional complex matrix algebras $M_n(\mathbb{C})$. In this case, the number n is called the *degree of* $\mathcal{A}$. Examples of Azumaya algebras include the sheaves of endomorphisms of complex vector bundles and the complex Clifford bundles $\mathbb{C}l(E)$ of oriented even-dimensional real vector bundles E. The Brauer group $\mathrm{Br}(X)$ classifies topological Azumaya algebras on X up to the usual Brauer equivalence: $\mathcal{A}_0$ and $\mathcal{A}_1$ are Brauer equivalent if there exist vector bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ such that there is an algebra isomorphism

$$\mathcal{A}_0 \otimes_{\mathcal{C}} \mathrm{End}(\mathcal{E}_0) \cong \mathcal{A}_1 \otimes_{\mathcal{C}} \mathrm{End}(\mathcal{E}_1).$$

Define $\mathrm{Br}(X)$ to be the free abelian group on isomorphism classes of Azumaya algebras modulo Brauer equivalence.

The group $\mathrm{Br}(X)$ is a subgroup of the cohomological Brauer group $\mathrm{Br}'(X) = \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$. This inclusion is constructed in the following way. There is an exact sequence

$$1 \to \mathbb{C}^* \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

which induces an exact sequence in non-abelian cohomology

$$\mathrm{H}^1(X, \mathrm{GL}_n) \to \mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, B\mathbb{C}^*) = \mathrm{H}^3(X, \mathbb{Z}).$$

The pointed set $\mathrm{H}^1(X, \mathrm{PGL}_n)$ classifies degree-n Azumaya algebras on X up to isomorphism whereas the left arrow sends an n-dimensional complex vector

bundle E to $\mathrm{End}(E)$. The map from the free abelian group on isomorphism classes of Azumaya algebras to $\mathrm{H}^3(X,\mathbb{Z})$ thus factors through the Brauer group.

By a result of Serre [25], every cohomological Brauer class $\alpha \in \mathrm{Br}'(X) = \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$ is represented by topological Azumaya algebras of varying degrees when X is a finite CW-complex, i.e., $\mathrm{Br}(X) = \mathrm{Br}'(X)$. The period-index problem is to determine which degrees arise for a given class, α. The index $\mathrm{ind}(\alpha)$ is defined to be the greatest common divisor of these degrees. The period $\mathrm{per}(\alpha)$ is the order of α in $\mathrm{Br}'(X)$. For any α, one has $\mathrm{per}(\alpha)|\,\mathrm{ind}(\alpha)$.

The Clifford bundle $\mathbb{C}l(E)$ associated to an oriented $2n$-dimensional real vector bundle E has class in the cohomological Brauer group given by $W_3(E)$, the third integral Stiefel-Whitney class of E. Thus, $W_3(E)$ is either period 1 or 2, depending on whether or not E supports a Spin^c-structure. The rank of $\mathbb{C}l(E)$ is 2^{2n}. If $\mathrm{per}(W_3(E)) = 2$, we find

$$2|\,\mathrm{per}(W_3(E))|\,\mathrm{ind}(W_3(E))|2^n.$$

To our knowledge, this period-index problem has not been considered before, although the parallel question in algebraic geometry has been the subject of a great deal of work. For instance, see [5, 14, 13, 17, 10, 39, 37, 39].

By analyzing the cohomology of the universal period r cohomological Brauer class, $K(\mathbb{Z}/r,2) \xrightarrow{\beta} K(\mathbb{Z},3)$, we show that the period and index have the same prime divisors, as in the algebraic case of fields. Then, using ideas from Antieau [4] we furnish upper bounds on $\mathrm{ind}(\alpha)$ depending only on $\mathrm{per}(\alpha)$ and the dimension of X.

By studying the cohomology of the projective unitary groups, we obtain obstructions to the representation of a class $\alpha \in \mathrm{Br}(X)$ by an Azumaya algebra of degree n. Using these obstructions, we construct families of examples of period r Brauer class whose indices form an unbounded sequence.

Suppose that X is a finite CW-complex with cohomological dimension d in the sense that $\mathrm{H}^n(X,A) = 0$ for all abelian sheaves A and all $n > d$. Let $\alpha \in \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$, and let $r = \mathrm{per}(\alpha)$. The classifying space $B\mathbb{Z}/r$ has finite stable homotopy in non-zero degrees. Let e_j^α denote the exponent of the finite abelian group $\pi_j^s(B\mathbb{Z}/r)$.

This is our main theorem.

Theorem 1.1 (theorem 4.1)**.**

$$\mathrm{ind}(\alpha)|\prod_{j=1}^{d-1} e_j^\alpha.$$

Now, consider the case where $\mathrm{per}(\alpha)$ is a prime ℓ. The theorem has the following corollary, which is obtained by the computation in [4] of the stable homotopy groups $\pi_j^s B\mathbb{Z}/\ell^k$ in a range.

Theorem 1.2 (theorem 4.6)**.** *Let X be a finite CW-complex of cohomological dimension d, let ℓ be a prime with $2\ell > d+1$, and let $\alpha \in \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$ satisfy $\mathrm{per}(\alpha) = \ell^k$. Then,*

$$\mathrm{ind}(\alpha)|\,\mathrm{per}(\alpha)^{[\frac{d}{2}]}.$$

By the Oka principle [24], which says that the topological and analytic classification of torsors of a complex Lie group agree on a Stein space, the same theorems hold for the analytic period-index problem on a Stein space having the homotopy type of a finite CW-complex.

To prove these theorems, we study the Atiyah-Hirzebruch spectral sequence for twisted K-theory $\mathrm{KU}(X)_\alpha$

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathbb{Z}(q/2)) \Rightarrow \mathrm{KU}^{p+q}(X)_\alpha.$$

Here twisted topological K-theory refers to the theory first introduced by Donovan-Karoubi [18], and then further studied by Rosenberg [45], Atiyah and Segal [7], and others. When X is compact, the index of α is also the generator of the image of the edge map

$$\mathrm{KU}^0(X)_\alpha \to \mathrm{H}^0(X, \mathbb{Z}).$$

We look for permanent cycles in $\mathrm{H}^0(X, \mathbb{Z})$; equivalently, we study the differentials leaving this group.

The study of such differentials is intricate. We should like to have a twisted analogue of the unit map $\pi_i^s \to \mathrm{KU}^{-i}$ but unfortunately the unit map cannot be twisted when α is non-trivial. Instead, if α is r-torsion, there is a spectrum, $\mathrm{S}[\mathbb{Z}/r]$, resembling a finite cover of the sphere spectrum, and a map $\mathrm{S}[\mathbb{Z}/r] \to \mathrm{KU}$ that extends the unit map and that may be twisted. We obtain from this a map $\mathrm{S}[\mathbb{Z}/r](X)_\beta \to \mathrm{KU}(X)_\alpha$, where β is a lift of α to $\mathrm{H}^2(X, \mathbb{Z}/r)$.

The utility of this map is that the homotopy groups of $\mathrm{S}[\mathbb{Z}/r]$ are all finite torsion groups and we know the torsion in low degrees (relative to r) by [4]. In the Atiyah-Hirzebruch spectral sequence for $\mathrm{S}[\mathbb{Z}/r]_\beta$, we obtain bounds on the differentials departing $\mathrm{H}^0(X, \mathbb{Z}) = \mathrm{H}^0(X, \mathrm{S}[\mathbb{Z}/r]^0)$. The result follows by considering a natural morphism of Atiyah-Hirzebruch spectral sequences. The idea of twisting the unit map is the innovation of [4].

To give lower bounds on the index, we consider when a map $\alpha : X \to K(\mathbb{Z}, 3)$ factors through $BPU_n \to K(\mathbb{Z}, 3)$, which is to say, when a class $\alpha \in \mathrm{H}^3(X, \mathbb{Z})$ may be represented by a degree-n Azumaya algebra. If such a factorization exists, then we obtain

$$\Omega X \to PU_n \xrightarrow{\sigma_n} K(\mathbb{Z}, 2).$$

The class σ_n in $\mathrm{H}^2(PU_n, \mathbb{Z})$ is studied in [9]. There, the order of σ_n^i is determined for all i. In particular, σ_n^{n-1} is non-zero, but $\sigma_n^n = 0$. This leads to a necessary condition for the factorization of $X \to K(\mathbb{Z}, 3)$ through BPU_n.

In another direction, the classifying space for r-torsion elements of $\mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ is the Eilenberg-MacLane space $K(\mathbb{Z}/r, 2)$ together with the Bockstein

$$\beta \in \mathrm{H}^3(K(\mathbb{Z}/r, 2), \mathbb{Z}),$$

which is order r. In order to use the Atiyah-Hirzebruch spectral sequence to study the period-index problem, it is crucial to understand the differentials in the β-twisted Atiyah-Hirzebruch spectral sequence for $K(\mathbb{Z}/r, 2)$. In particular, we want to study the differentials

$$d_{2k+1}^{\beta} : \mathrm{H}^0(K(\mathbb{Z}/r, 2), \mathbb{Z}) \dashrightarrow \mathrm{H}^{2k+1}(K(\mathbb{Z}/r, 2), \mathbb{Z}).$$

It is known [8] that $d_3^\beta(1) = \pm\beta$.

Atiyah-Segal [8] investigate the differentials for the twist of K-theory on $K(\mathbb{Q}, 3)$. In the torsion case, we show that there are infinitely many non-zero differentials leaving

$$\mathrm{E}_k^{0,0} \subseteq \mathrm{H}^0(K(\mathbb{Z}/r, 2), \mathbb{Z}).$$

In this case, of course, each of those differentials is torsion.

In fact, for a fixed prime l, we show that, after $d_3^\beta(1) = l$, one of the following $d_5^\beta(l), \ldots, d_{2l+1}^\beta(l)$ is non-zero in the β-twisted spectral sequence for $K(\mathbb{Z}/l, 2)$. Combining this with our results from the cohomology of the projective unitary groups, we find that $d_5^\beta(2)$ is non-zero in the β-twisted Atiyah-Hirzebruch spectral sequence for $K(\mathbb{Z}/2, 2)$.

In Antieau [4], the first named author used similar methods to define an étale index $\mathrm{eti}(\alpha)$ for classes of $\mathrm{Br}(X_{\text{ét}})$ and to give upper bounds in terms of the period. In that work, the question of whether the étale index ever differs from the period was not resolved. The examples of section 5.2 and the comparison map of theorem 2.11 give evidence that $\mathrm{eti} \neq \mathrm{per}$ as invariants on $\mathrm{Br}(X_{\text{ét}})$. We will consider the algebraization of these examples in a future paper.

The paper is organized as follows. In the rest of this section, we give background on the Brauer group and the period-index problem in various settings. In section 2 we establish the relevant technical tools in twisted K-theory we need, a comparison map from twisted étale K-theory to twisted topological K-theory, as well as the twisted unit map. In section 4, we prove the main general upper bounds. In section 5, after some diverting elementary number theory, we study the cohomology of the projective unitary spaces, and we apply this to furnish spaces with a class of period r and with arbitrarily large index.

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1.1 The Brauer group

For generalities on Azumaya algebras and Brauer groups on locally ringed sites, see Grothendieck [25]. For example, an Azumaya algebra on a topological space is a sheaf of algebras over the continuous complex-valued functions that is locally isomorphic to a matrix algebra over $\mathbb{C}$.

Throughout, X will denote a connected scheme, a complex analytic space, or a topological space. In any of these cases, there is a Brauer group $\mathrm{Br}(X)$ consisting of Brauer-equivalence classes of algebraic, analytic, or topological Azumaya algebras on X.

In each case, there is also a cohomological Brauer group, $\mathrm{Br}'(X)$, defined as $\mathrm{H}^2(X_{\text{ét}}, \mathbb{G}_m)_{\text{tors}}$ when X is algebraic, $\mathrm{H}^2(X, \mathcal{O}_X^*)_{\text{tors}}$ when X is a complex analytic space, and

$$\mathrm{H}^2(X, \mathcal{C}_X^*)_{\text{tors}} = \mathrm{H}^2(X, \mathbb{C}^*)_{\text{tors}} = \mathrm{H}^3(X, \mathbb{Z})_{\text{tors}}$$

when X is a topological space. There are natural inclusions

$$\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$$

in each case.

Given a complex algebraic scheme X, write $\mathrm{Br}(X_{\text{ét}})$ for the algebraic Brauer group of X, $\mathrm{Br}(X_{\text{an}})$ for the analytic Brauer group of X, and $\mathrm{Br}(X_{\text{top}})$ for the topological Brauer group, and similarly for Br'. There is a natural commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X_{\text{ét}}) \arrow[r] \arrow[d] & \mathrm{Br}(X_{\text{an}}) \arrow[r] \arrow[d] & \mathrm{Br}(X_{\text{top}}) \arrow[d] \\ \mathrm{Br}'(X_{\text{ét}}) \arrow[r] & \mathrm{Br}'(X_{\text{an}}) \arrow[r] & \mathrm{Br}'(X_{\text{top}}). \end{tikzcd}$$

Similarly, if X is a complex analytic space, let $\mathrm{Br}(X_{\text{an}})$ and $\mathrm{Br}(X_{\text{top}})$ be the analytic and topological Brauer groups, respectively. There is a natural commutative diagram

$$\begin{tikzcd} \mathrm{Br}(X_{\text{an}}) \arrow[r] \arrow[d] & \mathrm{Br}(X_{\text{top}}) \arrow[d] \\ \mathrm{Br}'(X_{\text{an}}) \arrow[r] & \mathrm{Br}'(X_{\text{top}}). \end{tikzcd}$$

If X is a complex algebraic scheme, and $\alpha \in \mathrm{Br}'(X_{\text{ét}})$, let α_{an} and α_{top} be the corresponding elements in the analytic and topological cohomological Brauer groups.

Example 1.3. If X is a complex K3 surface, then $\mathrm{Br}(X_{\text{ét}}) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho}$, where $0 \leq \rho \leq 20$ is the rank of the Neron-Severi group $\mathrm{NS}(X)$ [26, Section 3]. But, $\mathrm{Br}(X_{\text{top}}) = 0$ because $\mathrm{H}^3(X, \mathbb{Z}) = 0$.

Example 1.4. If X is a smooth projective rationally connected complex threefold, then $\mathrm{Br}(X_{\text{ét}}) = \mathrm{Br}(X_{\text{an}}) = \mathrm{Br}(X_{\text{top}})$, a finite abelian group, because $\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^3(X, \mathcal{O}_X) = 0$. Artin-Mumford [6] used the Brauer group to give some of the first examples of smooth projective unirational threefolds which are not rational. They found such a threefold with non-trivial 2-torsion in $\mathrm{Br}(X_{\text{top}})$. But, the topological Brauer group is a birational invariant of smooth projective complex varieties [6, Proposition 1], and $\mathrm{Br}(\mathbb{P}^n_{\text{top}}) = 0$.

1.2 Grothendieck's problem

Problem 1.5 (Grothendieck)**.** *Determine when* $\mathrm{Br}(X) \to \mathrm{Br}'(X)$ *is surjective.*

Here is a summary of known results:

- If X is a scheme having an ample family of line bundles, then Gabber (unpublished) and de Jong [16] have each shown that $\mathrm{Br}(X_{\text{ét}}) = \mathrm{Br}'(X_{\text{ét}})$. This is the case, for instance, for quasi-projective schemes over affine schemes.

- The map does not have to be a surjection for non-separated affine schemes, by an example of [21]. The example Q is an affine quadric cone glued to itself along the smooth locus. In this case, $\mathrm{Br}(Q_{\text{ét}}) = 0$, while $\mathrm{Br}'(Q_{\text{ét}}) = \mathbb{Z}/2$.

- The best result for complex spaces is the result of Schröer [47] which gives a purely topological condition that ensures $\mathrm{Br}(X_{\mathrm{an}}) = \mathrm{Br}'(X_{\mathrm{an}})$. This condition applies to complex Lie groups, Hopf manifolds, and all compact complex surfaces except for a class which conjecturally does not exist. These are the class VII surfaces that are not Hopf surfaces, Inoue surfaces, or surfaces containing a global spherical shell. Some special cases of Schröer's theorem had been obtained by Iversen [30], Hoobler [27], Berkovič [11], Elencwajg-Narasimhan [22], and Huybrechts-Schröer [29].

- Serre [25, Théorème 1.6] showed that if X has the homotopy type of a finite CW-complex, then
$$\mathrm{Br}(X_{\mathrm{top}}) = \mathrm{Br}'(X_{\mathrm{top}}) = \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}.$$
In particular, this holds for open sets in compact manifolds.

- By the Oka principle, the analytic and topological classification of PU_n-torsors is the same over a Stein space. Thus, if X is a (separated) Stein space having the homotopy type of a finite CW-complex, the result of Serre shows that
$$\mathrm{Br}(X_{\mathrm{an}}) = \mathrm{Br}'(X_{\mathrm{an}}).$$
In particular, this holds for Stein submanifolds of compact complex manifolds.

- As a negative example, if $X = K(\mathbb{Z}/m, 2)$, then $\mathrm{Br}(X_{\mathrm{top}}) = 0$, while
$$\mathrm{Br}'(X_{\mathrm{top}}) = \mathbb{Z}/m.$$

1.3 The prime divisors problem

Definition 1.6. For any class $\alpha \in \mathrm{Br}'(X)$, the period of α, denoted $\mathrm{per}(\alpha)$, is the order of α in the group $\mathrm{Br}'(X)$.

In general, the degree of an Azumaya algebra $\mathcal{A}$ is the positive square-root of its rank as an $\mathcal{O}_X$-module. This is a locally constant integer, and hence a class in $\mathrm{H}^0(X, \mathbb{Z})$. Henceforth, for simplicity, assume that X is connected, so that $\deg(\mathcal{A}) \in \mathbb{Z}$. It is a general fact that if the class of $\mathcal{A}$ is $\alpha \in \mathrm{Br}(X)$, then
$$\mathrm{per}(\alpha) | \deg(\mathcal{A}).$$
Indeed, A determines a PGL_n-torsor, where $n = \deg(\mathcal{A})$. The class $\alpha \in \mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$ is the coboundary of this torsor
$$\mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbb{G}_m),$$

and this factors through

$$\mathrm{H}^2(X, \mu_n) \to \mathrm{H}^2(X, \mathbb{G}_m),$$

assuming that n is prime to the characteristics of the residue fields of X. For details, see [25].

Definition 1.7. If $\alpha \in \mathrm{Br}(X)$, define the index of α to be

$$\mathrm{ind}(\alpha) = \gcd\{\deg(\mathcal{A}) : \mathcal{A} \in \alpha\}.$$

If $\alpha \in \mathrm{Br}'(X) \setminus \mathrm{Br}(X)$, set $\mathrm{ind}(\alpha) = +\infty$.

In general, $\mathrm{per}(\alpha) | \mathrm{ind}(\alpha)$.

Problem 1.8. *Do* $\mathrm{per}(\alpha)$ *and* $\mathrm{ind}(\alpha)$ *have the same prime divisors?*

Known results:

- If k is a field, $\alpha \in \mathrm{Br}(k) = \mathrm{Br}'(k)$, then the period and index of α have the same prime divisors.
- If $X = \mathrm{Spec}\, A$ is an affine scheme, then Gabber has shown that $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ have the same prime divisors.
- An unpublished argument of Saltman using extension of coherent sheaves shows that the period and index have the same prime divisors for Brauer classes on regular noetherian irreducible schemes.

We prove below, in corollary 3.2, that if X is a finite CW-complex, and if $\alpha \in \mathrm{Br}(X)$, then $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ have the same prime divisors.

1.4 The period-index problem

A well-known problem for fields, where $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ have the same prime divisors, is to bound the smallest integers $e(\alpha)$ such that $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{e(\alpha)}$.

Conjecture 1.9 (Colliot-Thélène)**.** *Let k be either a C_d-field or the function field of a d-dimensional variety over an algebraically closed field. Let $\alpha \in \mathrm{Br}(k)$, and suppose that* $\mathrm{per}(\alpha)$ *is prime to the characteristic of k. Then,*

$$\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{d-1}$$

Results:

- If S is a surface over an algebraically closed field k, and $\alpha \in \mathrm{Br}(k(S))$, then de Jong [17] showed that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.
- If S is a surfaces over $\mathbb{F}_q$, and if $\alpha \in \mathrm{Br}(\mathbb{F}_q(S))$, then $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^2$, by Lieblich [38].

- If C is a curve over $\mathbb{Q}_p$, and if $\alpha \in \mathrm{Br}(\mathbb{Q}_p(C))$, then $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^2$, by Saltman [46].

A few other results are known, but higher dimensional cases remain elusive.

Problem 1.10 (Period-index)**.** *Fix X, and find an integer e_r such that*

$$\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{e_r}$$

for all $\alpha \in {}_{r^\infty}\mathrm{Br}(X)$, where ${}_{r^\infty}\mathrm{Br}(X)$ denotes the r-primary part of the Brauer group.

The main global result of which we are aware is for surfaces over algebraically closed fields, where per $=$ ind by Lieblich [39]. Although, the argument of Saltman mentioned above reduces the problem for smooth varieties over algebraically closed fields to their function fields.

It is worth noting here the example of Kresch [35] which uses Hodge theory and singular cohomology to give period 2 classes on certain complex threefolds which are not representable by quaternion algebras. Our topological period-index theorem is orthogonal to his example, as the classes he constructs vanish in $\mathrm{H}^3(X, \mathbb{Z})$.

Our theorems 4.1 and 4.6 give a solution to this problem for finite CW-complexes. In dimension 4 a stronger bound than ours is possible via a direct argument using the Atiyah-Hirzebruch spectral sequence.

2 Twisted K-theory

In this section, we recall the twisted topological complex K-theory spectrum $\mathrm{KU}(X)_\alpha$ for any space X and any class $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$. This spectrum is given by Atiyah-Segal [7, Section 4], generalizing work of Donovan and Karoubi [18]. For X a CW-complex and α in the Brauer group, we introduce the twisted algebraic K-theory spectrum $\mathcal{K}^\alpha(X)$, which is the algebraic K-theory of the category of α-twisted topological vector bundles on X. There is a map of spectra $\mathcal{K}^\alpha(X) \to \mathrm{KU}(X)_\alpha$, which induces an isomorphism $\mathcal{K}_0^\alpha(X) \xrightarrow{\cong} \mathrm{KU}^0(X)_\alpha$ when X is compact. When X is a complex noetherian scheme, we obtain in this way

$$\mathbf{K}^\alpha(X) \to \mathcal{K}^\alpha(X) \to \mathrm{KU}(X)_\alpha,$$

where $\mathbf{K}^\alpha(X)$ is the algebraic K-theory of locally free and finite rank α-twisted $\mathcal{O}_X$-modules on X. Since $\mathrm{KU}(X)_\alpha$ satisfies descent, the map above produces a map of spectra

$$a_{\text{ét}}\mathbf{K}^\alpha(X) \to \mathrm{KU}(X)_\alpha,$$

where $a_{\text{ét}}\mathbf{K}^\alpha$ is the étale-sheafification of the presheaf of spectra $\mathbf{K}^\alpha$.

Having dealt with the foundational questions to our satisfaction, we twist the unit map $SS \to \mathrm{KU}$ following [4]. Finally, we use the rank map for α-twisted K-theory to define an approximation to the index of α. If X is a finite CW-complex, this approximation is precisely the index, which allows us to use

the twisted K-theory to settle the prime-divisor problem and, when combined with the twisted-unit map, to provide estimates for the period-index problem. These applications will be the content of the following two sections.

2.1 Twists of KU

Definition 2.1. Let $\alpha \in \mathrm{H}^3(X, \mathbb{Z}) \cong \mathrm{H}^1(X, \mathrm{PU}(\mathcal{H}))$. Let $\mathrm{KU}(X)_\alpha$ be the twisted K-theory spectrum constructed in Atiyah-Segal [7], and let $\mathrm{KU}(\cdot)_\alpha$ be the contravariant functor on spaces mapping to X given by

$$V \mapsto \mathrm{KU}(V)_{\alpha|_V}.$$

Note that there is a sign ambiguity involved in the isomorphism $\mathrm{H}^3(X, \mathbb{Z}) \cong \mathrm{H}^1(X, \mathrm{PU}(\mathcal{H}))$. This ambiguity arises from the two different actions of $\mathrm{PU}(\mathcal{H})$ on the space $U(\mathcal{H})$, either by $A \cdot B = A^{-1}BA$ or by $A \cdot B = ABA^{-1}$. The associated twisted algebras of Fredholm operators are opposite each other. We choose the following normalization: use the construction that results in $d_3^\alpha(1) = \alpha$ in the Atiyah-Hirzebruch spectral sequence for twisted KU-theory (see [8, Proposition 4.6]).

This sign ambiguity has no bearing on the period-index problem of this paper. The methods will always produce a positive integer, the index, that is the same for α and $-\alpha$.

We use the following properties of $\mathrm{KU}(\cdot)_\alpha$.

Proposition 2.2. *The presheaf of spectra $\mathrm{KU}(\cdot)_\alpha$ commutes with homotopy colimits (taken in spaces) of spaces over X. In particular, if $\mathcal{V}_* \to X$ is a hypercover of X, then*

$$\mathrm{KU}(X)_\alpha \xrightarrow{\sim} \operatorname*{holim}_{\Delta} \mathrm{KU}(\mathcal{V}_*)_\alpha.$$

That is, $\mathrm{KU}(\cdot)_\alpha$ satisfies descent.

Proof. Suppose that X is a CW-complex, and that $\{P_n\}_{n\in\mathbb{Z}}$ is a spectrum over X. It suffices here for this to mean simply that there are fibrations $\pi_n : P_n \to X$ with fixed distinguished sections and fixed weak equivalences $\Omega_X P_n \simeq_X P_{n-1}$, where the weak equivalence induces weak equivalences on the fibers over X. So, P_n is an Ω-spectrum over X. Atiyah and Segal [7, Section 4] show that twisted K-theory is represented by just such a spectrum. To prove the proposition, it suffices to show that the sections of the spectrum over a space $Y \to X$ form an Ω-spectrum, and to show that taking spaces of sections commutes with homotopy colimits.

Now, suppose that $i \mapsto Y_i$ is an I-diagram of CW-complexes, each mapping to X, and suppose that $\mathrm{hocolim}_i\, Y_i \to X$ is a weak equivalence.

For an arbitrary CW-complex W let $\mathbf{Map}_\pi(W, P_n)$ denote the mapping

spaces over π, namely the pull-back

$$\begin{tikzcd}
\mathbf{Map}_\pi(W, P_n) \arrow[r] \arrow[d] & \mathbf{Map}(W, P_n) \arrow[d] \\
* \arrow[r] & \mathbf{Map}(W, X).
\end{tikzcd}$$

which is also a homotopy pull-back since $P_n \to X$ is a fibration. There is a natural homotopy equivalence $\operatorname{holim}_i \mathbf{Map}(Y_i, Z) \to \mathbf{Map}(X, Z)$ for an arbitrary space Z; by application of $\operatorname{holim}_i(Y_i, \cdot)$ to the diagram

$$\begin{tikzcd}
& P_n \arrow[d] \\
* \arrow[r] & X
\end{tikzcd}$$

and use of the 'Fubini' theorem for homotopy limits, we deduce that

$$\mathbf{Map}_\pi(X, P_n) \to \operatorname*{holim}_i \mathbf{Map}_\pi(Y_i, P_n)$$

is a weak equivalence. Now, it suffices to prove that

$$\mathbf{Map}_\pi(X, \Omega_X P_n) \simeq \Omega \mathbf{Map}_\pi(X, P_n).$$

This also follows from the fact that $\Omega_X P_n$ is also a limit.

This implies that $\mathrm{KU}(\cdot)$ satisfies descent since $\operatorname{hocolim}_\Delta(V_*) \simeq X$ by [20]. □

Proposition 2.3. *Let X be a topological space, and let $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$. Then, there is a descent spectral sequence (Atiyah-Hirzebruch spectral sequence)*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathbb{Z}(q/2)) \Rightarrow \mathrm{KU}^{p+q}(X)_\alpha$$

with differentials d_r^α of degree $(r, -r+1)$, which converges strongly if X is a finite dimensional CW-complex. The edge map $\mathrm{KU}^0(X)_\alpha \to \mathrm{H}^0(X, \mathbb{Z})$ is the index map (or rank map, or reduced norm map).

Proof. This is [7, Theorem 4.1]. □

We will actually use a re-indexed (Bousfield-Kan style) version of this spectral sequence:

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \pi_q \mathrm{KU}) \Rightarrow \pi_{q-p} \mathrm{KU}(X)_\alpha.$$

Now, the differentials are d_r^α of degree $(r, r-1)$.

Proposition 2.4. *The differential*

$$d_3^\alpha : \mathrm{H}^0(X, \mathbb{Z}) \to \mathrm{H}^3(X, \mathbb{Z})$$

sends 1 to α.

Proof. This is [8, Proposition 4.6], where we have altered the construction to change the sign. See also Antieau [3] for the analogous computation in twisted algebraic K-theory. □

2.2 Twisted algebraic K-theory

Throughout, if $\mathcal{A}$ is an Azumaya algebra over X, an $\mathcal{A}$-module will mean a left $\mathcal{A}$-module whose total space is a finite dimensional complex topological vector bundle over X. The category of $\mathcal{A}$-modules and $\mathcal{A}$-module homomorphisms will be denoted $\mathrm{Vect}^{\mathcal{A}}$. The category $\mathrm{Vect}^{\mathcal{A}}$ is a topological category with direct sum, so by Segal [48] it has an algebraic K-theory spectrum $\mathcal{K}^{\mathcal{A}}(X)$.

If $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$, there is also a category Vect^{α} of α-twisted finite dimensional complex vector bundles (see for instance [15] or [34]). Let $\mathcal{K}^{\alpha}(X)$ be the K-theory of this topological category with direct sum. If $\mathcal{E}$ is an α-twisted vector bundle, then its sheaf of endomorphisms is an Azumaya with class α. The following statement gives the converse.

Proposition 2.5. *If $\mathcal{A}$ is an Azumaya algebra of degree n with class α, then $\mathcal{A} \cong \mathrm{End}(\mathcal{E})$ for some α-twisted vector bundle $\mathcal{E}$ of rank n. The α-twisted sheaf is unique up to tensoring with (non-twisted) line bundles.*

Proof. See [15, Theorem 1.3.5] or [34, Section 3]. □

Proposition 2.6. *Tensoring with $\mathcal{E}^*$, the dual of $\mathcal{E}$, induces an equivalence of categories $\mathrm{Vect}^{\alpha} \to \mathrm{Vect}^{\mathcal{A}}$.*

Proof. See [15, Theorem 1.3.7] or [34, Section 3]. □

Proposition 2.7. *If $\mathcal{A}$ and $\mathcal{B}$ are Brauer-equivalent Azumaya algebras over X, then $\mathrm{Vect}^{\mathcal{A}}$ and $\mathrm{Vect}^{\mathcal{B}}$ are equivalent categories.*

Proof. This follows from the previous two propositions. □

If X is a finite CW-complex, the twisted topological K-group $\mathrm{KU}^0(X)_{\alpha}$ may be identified with the Grothendieck group of left $\mathcal{A}$-modules.

Proposition 2.8. *If X is compact and Hausdorff, and if $\mathcal{A}$ is an Azumaya algebra on X with class α (so that α is torsion), then $\mathcal{K}_0^{\mathcal{A}}(X) \cong \mathrm{KU}^0(X)_{\alpha}$. This isomorphism is uniquely defined up to the natural action of $\mathrm{H}^2(X, \mathbb{Z})$ on the left.*

Proof. This follows from [7, Section 3.1]. In fact, they show that $\mathrm{KU}^0(X)_{\alpha}$ is (isomorphic to) the Grothendieck group of finitely generated projective left $\Gamma(X, \mathcal{A})$-modules. The proof of Swan's theorem [50] extends to show that the categories of finitely generated projective left $\Gamma(X, \mathcal{A})$-modules and left $\mathcal{A}$-modules that are finite dimensional vector bundles are equivalent. See also Karoubi [34, Section 4, Section 8.3]. □

Corollary 2.9. *If X is a finite CW-complex and if $\mathcal{A}$ is an Azumaya algebra on X, there is a natural map $\mathcal{K}^{\mathcal{A}}(X) \to \mathrm{KU}(X)_{\alpha}$ inducing an isomorphism in degree zero. This map is uniquely defined up to the action of $\mathrm{H}^2(X, \mathbb{Z})$ on the left.*

Proof. The isomorphism of the proposition is induced by a map from the classifying space of the topological category $\mathrm{Vect}^{\mathcal{A}}$ to the zero-space of $\mathrm{KU}(X)_\alpha$. See the proof of [7, Definition 3.4]. By the standard adjunction between Γ-spaces and spectra in Segal [48], this induces a map $\mathcal{K}^{\mathcal{A}}(X) \to \mathrm{KU}(X)_\alpha$. □

Proposition 2.10. *If X is a countable CW-complex, and if $\mathcal{A}$ is an Azumaya algebra with class $\alpha \in \mathrm{Br}(X_{\mathrm{top}})$, then there is a map of spectra*

$$\mathcal{K}^{\mathcal{A}}(X) \to \mathrm{KU}(X)_\alpha.$$

This map is uniquely defined up to the natural action of $\mathrm{H}^2(X, \mathbb{Z})$ on the left.

Proof. First we give the proof for countable d-dimensional CW-complexes X. Suppose that we have constructed maps $\mathcal{K}^{\mathcal{A}}(X) \to \mathrm{KU}(X)_\alpha$ for all finite CW-subcomplexes of X and all countable $(d-1)$-dimensional CW-subcomplexes of X. Let X be a countable d-dimensional CW-complex, let X_{d-1} be its $(d-1)$-skeleton, and let

$$X_{d-1} = X_{d,0} \subseteq X_{d,1} \subseteq X_{d,2} \subseteq \cdots = X$$

be an inductive construction of X, where $X_{d,k}$ is constructed from $X_{d,k-1}$ by attaching a single cell. Assume we have constructed $\mathcal{K}^{\mathcal{A}}(X_{d,k}) \to \mathrm{KU}(X_{d,k})_\alpha$. Then, $X_{d,k+1}$ is the homotopy pushout

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & X_{d,k} \arrow[d] \\ D^d \arrow[r] & X_{d,k+1}. \end{tikzcd}$$

Since $\mathrm{KU}(\cdot)_\alpha$ commutes with homotopy colimits, we get a homotopy pull-back diagram of spectra by proposition 2.2

$$\begin{tikzcd} \mathrm{KU}(X_{d,k+1})_\alpha \arrow[r] \arrow[d] & \mathrm{KU}(D^d)_\alpha \arrow[d] \\ \mathrm{KU}(X_{d,k})_\alpha \arrow[r] & \mathrm{KU}(S^{d-1})_\alpha. \end{tikzcd}$$

By the induction hypothesis, $K^{\mathcal{A}}(X_{d,k+1})$ fits into a homotopy commutative diagram

$$\begin{tikzcd} \mathcal{K}^{\mathcal{A}}(X_{d,k+1}) \arrow[r] \arrow[d] & \mathrm{KU}(D^d)_\alpha \arrow[d] \\ \mathrm{KU}(X_{d,k})_\alpha \arrow[r] & \mathrm{KU}(S^{d-1})_\alpha. \end{tikzcd}$$

So, by the property of the homotopy pull-back, we get a map

$$\mathcal{K}^{\mathcal{A}}(X_{d,k+1}) \to \mathrm{KU}(X_{d,k+1})_\alpha.$$

Inductively, we get a map of sequences of spectra

$$\begin{tikzcd} \mathcal{K}^{\mathcal{A}}(X_{d,k}) \arrow[r] \arrow[d] & \mathrm{KU}(X_{d,k})_\alpha \arrow[d] \\ \mathcal{K}^{\mathcal{A}}(X_{d,k-1}) \arrow[r] & \mathrm{KU}(X_{d,k-1})_\alpha. \end{tikzcd}$$

Again, $\mathcal{K}^{\mathcal{A}}(X)$ then maps to the homotopy limit of the tower on the right. By proposition 2.2, this is $\mathrm{KU}(X)_\alpha$. The case where X is countable and infinite dimensional is proved in the same way. ◻

2.3 Comparison maps

Let X be a noetherian complex scheme, and let $\alpha \in \mathrm{Br}(X_{\text{ét}})$. Let $a_{\text{ét}}\mathbf{K}^\alpha$ be the étale sheafification of the α-twisted algebraic K-theory presheaf $\mathbf{K}^\alpha$, as studied in Antieau [4]. Let $\mathrm{KU}(X)_\alpha$ be the twisted topological K-theory associated to the image of α by $\mathrm{Br}(X_{\text{ét}}) \to \mathrm{Br}(X_{\text{top}})$.

Theorem 2.11. *There is a map of hypersheaves of spectra*

$$a_{\text{ét}}\mathbf{K}^\alpha \to \mathrm{KU}(\cdot)_\alpha$$

on the small étale site of X, and this map induces an isomorphism on 0-homotopy sheaves $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$. This map is unique up to the action on the left of $\mathrm{Pic}(X)$.

Proof. Let $\mathcal{A}$ be an algebraic Azumaya algebra with class $\alpha \in \mathrm{Br}(X_{\text{ét}})$. Since any open subset of X is triangulable, there is a map of presheaves of spectra on the small étale site of X

$$\mathcal{K}^{\mathcal{A}} \to \mathrm{KU}(\cdot)_\alpha$$

by proposition 2.10, where $\mathcal{K}^{\mathcal{A}}(U)$ is the algebraic K-theory of topological left $\mathcal{A}$-modules which are vector bundles over U. There is a map of presheaves of spectra

$$\mathbf{K}^{\mathcal{A}} \to \mathcal{K}^{\mathcal{A}},$$

where $\mathbf{K}^{\mathcal{A}}$ is the presheaf of K-theory spectra of algebraic $\mathcal{A}$-modules which are locally free and finite rank as coherent $\mathcal{O}_X$-modules. Since $\mathrm{KU}(\cdot)_\alpha$ satisfies descent for hypercovers by proposition 2.2, it follows by [19] that the map

$$\mathbf{K}^{\mathcal{A}} \to \mathcal{K}^{\mathcal{A}} \to \mathrm{KU}(\cdot)_\alpha$$

factors through the sheafification $a_{\text{ét}}\mathbf{K}^{\mathcal{A}}$ of $\mathbf{K}^\alpha$. By [39], [15], or [4], there are natural weak equivalences of presheaves $\mathbf{K}^\alpha \simeq \mathbf{K}^{\mathcal{A}}$ unique up to an action of $\mathrm{Pic}(X)$ on the left. The theorem follows. ◻

2.4 Twisting units

This section is technical, but it provides the key tool for our main period-index result: the twisted unit morphism.

Let S be the sphere spectrum. In [42, Corollary 6.3.2.16], drawing on work of [23], [28], and others, Lurie constructs a symmetric monoidal category structure on $\mathrm{Mod}_{\mathrm{S}}$, the ∞-category of S-modules (or, equivalently, the ∞-category of spectra). The associative algebras in $\mathrm{Mod}_{\mathrm{S}}$ are shown to model A_∞-ring spectra, and the commutative algebras in $\mathrm{Mod}_{\mathrm{S}}$ are shown to model E_∞-ring spectra (see the introduction to Section 7.1 of [42]).

Let R be a commutative S-algebra (an algebra object in $\mathrm{Mod}_{\mathrm{S}}$), and let $\mathrm{Line}_{\mathrm{R}}$ denote the ∞-groupoid in $\mathrm{Mod}_{\mathrm{R}}$ generated by R-modules equivalent to R. Finally, set $BGL_1\mathrm{R} = |\mathrm{Line}_{\mathrm{R}}|$, the geometric realization. To a map of spaces $f : X \to BGL_1\mathrm{R}$, one associates a homotopy class of maps of simplicial sets $f : \mathrm{Sing}(X) \to \mathrm{Line}_{\mathrm{R}}$, where $\mathrm{Sing}(X)$ is the simplicial set of simplices in X. This should be thought of as a locally-free rank one line bundle over the constant sheaf R on X.

Given $f : \mathrm{Sing}(X) \to \mathrm{Line}_{\mathrm{R}}$, in [1], an R-module X^f, called the Thom spectrum of f, is defined as

$$X^f = \mathrm{colim}(\mathrm{Sing}(X) \to \mathrm{Line}_{\mathrm{R}} \to \mathrm{Mod}_{\mathrm{R}}).$$

See also [2]. This is the colimit of the $\mathrm{Sing}(X)$-diagram f in the ∞-category $\mathrm{Mod}_{\mathrm{R}}$. See [40, Section 1.2.13 and Chapter 4] for definitions and properties of these colimits. In [40, Section 4.2.4], Lurie shows that these colimits agree with the usual notion of homotopy colimits when the diagram takes values in the nerve of a simplicial model category.

Definition 2.12. The f-twist of R is the internal dual to X^f:

$$\mathrm{R}(X)_f = \mathrm{Mod}_{\mathrm{R}}(X^f, \mathrm{R}).$$

Let $\mathrm{R}(\cdot)_f$ be the associated presheaf of spectra on X.

Example 2.13. The space $BGL_1\,\mathrm{KU}$ is equivalent to

$$K(\mathbb{Z}/2, 1) \times K(\mathbb{Z}, 3) \times BB\,\mathrm{SU}\,.$$

So, classes in $\mathrm{H}^3(X, \mathbb{Z})$ give twisted K-theory spectra. It is explained in [2, Section 5] how this agrees with the Atiyah-Segal construction.

Let $\mathbf{Shv}_{\mathrm{Sp}}(X)^\wedge$ denote the ∞-category of hypersheaves of spectra on X. This category can be constructed from presheaves of spectra on X by taking the full subcategory of presheaves which are local with respect to ∞-connective maps on X. See [41, Section 1] and [40, Lemma 6.5.2.12]. For another perspective, see [31]. These are related by [40, Proposition 6.5.2.14].

Example 2.14. By proposition 2.2, $\mathrm{KU}(\cdot)_\alpha$ is a hypersheaf of spectra.

In general, if $u : \mathrm{R} \to \mathrm{T}$ is a map of commutative S-algebras, and if $f : X \to BGL_1\mathrm{T}$ is a map, it is not necessarily possible to find an R-module M in $\mathbf{Shv}_{\mathrm{Sp}}(X)^\wedge$ together with a map $\mathrm{M} \to \mathrm{R}(\cdot)_f$ restricting to u on small open sets. For instance, this is never the case for the unit $u : \mathrm{S} \to \mathrm{KU}$ and the non-trivial twists $\mathrm{KU}(\cdot)_\alpha$ for $\alpha \in \mathrm{H}^3(X, \mathbb{Z})$.

However, we show now, that if $\alpha \in \mathrm{H}^3(X, \mathbb{Z})$ is torsion, then there is a finite cover of S, say T, a map $u : \mathrm{T} \to \mathrm{KU}$ extending $\mathrm{S} \to \mathrm{KU}$, and a twist, $u_\beta : \mathrm{T}(\cdot)_\beta \to \mathrm{KU}(\cdot)_\alpha$ derived from a map $\beta : X \to BGL_1\mathrm{T}$.

Definition 2.15. Let $\mathrm{S}[\mathbb{Z}/r]$ denote be the algebraic K-theory of the symmetric monoidal category $rSets$ of finite disjoint unions of $\mathbb{Z}/r$ with the faithful $\mathbb{Z}/r$ action and $\mathbb{Z}/r$-equivariant maps. The spectrum $\mathrm{S}[\mathbb{Z}/r]$ is a commutative S-algebra.

Lemma 2.16. *The homotopy groups of* $\mathrm{S}[\mathbb{Z}/r]$ *are*

$$\pi_q \mathrm{S}[\mathbb{Z}/r] \cong \pi_q^s \oplus \pi_q^s(B\mathbb{Z}/r),$$

where $B\mathbb{Z}/r$ is the classifying space of $\mathbb{Z}/r$. Moreover, $\pi_q^s(B\mathbb{Z}/r)$ surjects onto the r-primary part of π_q^s.

Proof. The first statement follows from the Barratt-Kahn-Priddy-Quillen theorem (see Thomason [52, Lemma 2.5]), which says that $\mathrm{S}[\mathbb{Z}/m]$ is equivalent to the spectrum $\Sigma^\infty(B\mathbb{Z}/m)_+$. The second statement follows from the Kahn-Priddy theorem [32, 33]. □

Lemma 2.17. *For any r, the unit map* $\mathrm{S} \to \mathrm{KU}$ *factors through* $\mathrm{S} \to \mathrm{S}[\mathbb{Z}/r]$*. And, the map* $\mathrm{S}[\mathbb{Z}/r] \to \mathrm{KU}$ *is a map of* S*-algebras.*

Proof. This follows by embedding $\mathbb{Z}/r$ into $\mathbb{C}^*$ as the rth roots of unity and then applying Segal's construction [48] to obtain $\mathrm{S} \to \mathrm{S}[\mathbb{Z}/r] \to \mathrm{ku}$. Now, compose with $\mathrm{ku} \to \mathrm{KU}$. That KU is in fact a commutative S-algebra is [23, Theorem 4.3]. □

Proposition 2.18. *There is a natural map*

$$K(\mathbb{Z}/r, 2) \to BGL_1\mathrm{S}[\mathbb{Z}/r]$$

Proof. The group $\mathbb{Z}/r$ acts naturally as monoidal auto-equivalences of the symmetric monoidal category $rSets$. So, $\mathbb{Z}/r$ acts naturally on $\mathrm{S}[\mathbb{Z}/r]$. This gives a map $B\mathbb{Z}/r \to GL_1\mathrm{S}[\mathbb{Z}/r]$. By delooping, we get the desired map. □

Proposition 2.19. *Let $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ be r-torsion, and lift α to $\beta \in \mathrm{H}^2(X, \mathbb{Z}/r)$. Then, there is a natural map*

$$\mathrm{S}[\mathbb{Z}/r](\cdot)_\beta \to \mathrm{KU}(\cdot)_\alpha$$

of presheaves of spectra restricting locally to the map of lemma 2.17.

Proof. It suffices to check that the deloopings

$$\begin{tikzcd} K(\mathbb{Z}/r,1) \arrow[r] \arrow[d, "\beta"'] & GL_1\mathrm{S}[\mathbb{Z}/r] \arrow[d] \\ K(\mathbb{Z},2) \arrow[r] & GL_1\,\mathrm{KU} \end{tikzcd}$$

commute. And, to check this we can check on maps out of finite CW-complexes. So, let $\gamma \in \mathrm{H}^1(X,\mathbb{Z}/r)$ be a $\mathbb{Z}/r$-torsor. The corresponding automorphism of the constant sheaf $\mathrm{S}[\mathbb{Z}/r]$ is given by tensoring with γ. And the induced automorphism of the constant sheaf KU is tensoring with the complex line bundle induced by γ. On the other hand $\mathrm{H}^1(X,\mathbb{Z}/r) \to \mathrm{H}^2(X,\mathbb{Z})$ sends γ to $c_1(\mathcal{L})$, where $\mathcal{L}$ is the complex line bundle associated to γ. The corresponding automorphism of KU is given by tensoring with $\mathcal{L}$, by construction [2]. □

Corollary 2.20. *Let $\alpha \in \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$ be r-torsion, and lift α to $\beta \in \mathrm{H}^2(X,\mathbb{Z}/r)$. Then, there is a spectral sequence*

$$\mathrm{E}^2_{p,q} = \mathrm{H}^p(X,\pi_q\mathrm{S}[\mathbb{Z}/r]) \Rightarrow \pi_{q-p}\mathrm{S}[\mathbb{Z}/r]^\wedge(X)_\beta,$$

where $\mathrm{S}[\mathbb{Z}/r]^\wedge(\cdot)_\beta$ is the hypersheafification of the presheaf $\mathrm{S}[\mathbb{Z}/r](\cdot)_\beta$. The differentials d^β_k are of degree $(k,k-1)$. There is a morphism of spectral sequences

$$\Big(\mathrm{H}^p(X,\pi_q\mathrm{S}[\mathbb{Z}/r]) \Rightarrow \pi_{q-p}\mathrm{S}[\mathbb{Z}/r]^\wedge(X)_\beta\Big) \to \Big(\mathrm{H}^p(X,\pi_q\,\mathrm{KU}) \Rightarrow \pi_{q-p}\,\mathrm{KU}(X)_\alpha\Big).$$

Proof. This is the standard descent spectral sequence. Note that $\mathrm{S}[\mathbb{Z}/r](\cdot)_\beta \to \mathrm{KU}(\cdot)_\alpha$ factors through hypersheafification $\mathrm{S}[\mathbb{Z}/r](\cdot)_\beta \to \mathrm{S}[\mathbb{Z}/r]^\wedge(\cdot)_\beta$ because $\mathrm{KU}(\cdot)_\alpha$ is a hypersheaf by proposition 2.2. □

2.5 The K-theoretic index

There are rank maps forming a commutative diagram

$$\begin{tikzcd} \mathcal{K}^\alpha_0(X) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, equal] \\ \mathrm{KU}^0(X)_\alpha \arrow[r] & \mathbb{Z} \end{tikzcd}$$

The index $\mathrm{ind}(\alpha)$ is the positive generator of the image of the top arrow, and we define the **K**-*index* $\mathrm{ind}_{\mathbf{K}}(\alpha)$ to be the positive generator of the image of the bottom arrow. We may view the rank map as the pull-back of an α-twisted virtual bundle to a point, that is to say the bottom map may be written as $\mathrm{KU}^0(X)_\alpha \to \mathrm{KU}^0(*)_0 \cong \mathbb{Z}$. We may compare the Atiyah-Hirzebruch spectral sequences for the twisted KU-theory of X and $*$ to arrive at the following conclusion.

Proposition 2.21. *Let X be a finite-dimensional connected CW-complex and let $\alpha \in \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}$. Then $\mathrm{ind}_{\mathbf{K}}(\alpha)$ is the positive generator of the subgroup $\mathrm{E}^{0,0}_\infty$ of the group $\mathrm{E}^{0,0}_2 = H^0(X,\mathbb{Z}) = \mathbb{Z}$ in the Atiyah-Hirzebruch spectral sequence of proposition 2.3.*

Note that under $\mathcal{K}_0^{\mathcal{A}}(X) \cong \mathcal{K}_0^{\alpha}(X)$, the map $\mathcal{K}_0^{\mathcal{A}}(X) \to \mathbb{Z}$ is the reduced norm map. It sends an $\mathcal{A}$-module to the quotient of its rank as a complex vector bundle by the degree of $\mathcal{A}$.

Lemma 2.22. *For any* $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$*,*

$$\mathrm{per}(\alpha) | \mathrm{ind}_{\mathbf{K}}(\alpha) | \mathrm{ind}(\alpha).$$

Moreover, ind(α) *is equal to*

$$\gcd\{\deg(\mathcal{A}) | [\mathcal{A}] = \alpha\}.$$

Proof. The statement about divisibilities follows from the commutative diagram. The second statement follows from propositions 2.5 and 2.6. ☐

Lemma 2.23. *If X is a finite CW-complex, then* $\mathrm{ind}_{\mathbf{K}}(\alpha) = \mathrm{ind}(\alpha)$.

Proof. This follows immediately from proposition 2.8. ☐

Example 2.24. Suppose that X is a finite CW-complex of dimension at most 6, and suppose that $H^5(X, \mathbb{Z})$ is torsion-free. Let $\alpha \in \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$. Then, $\mathrm{ind}(\alpha) = \mathrm{ind}_{\mathbf{K}}(\alpha) = \mathrm{per}(\alpha)$. It suffices to compute $d_5^{\alpha}(\mathrm{per}(\alpha))$. This lands in a subquotient of $H^5(X, \mathbb{Z})$. The differential

$$d_3^{\alpha} : \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^5(X, \mathbb{Z})$$

is zero. Indeed, the differential is torsion since α is torsion, so it has torsion image. But, $\mathrm{H}^5(X, \mathbb{Z})$ is non-torsion. Thus, as there are no differentials leaving $\mathrm{H}^5(X, \mathbb{Z})$, $\mathrm{E}_5^{5,-4} = \mathrm{H}^5(X, \mathbb{Z})$. Then, $d_5^{\alpha}(\mathrm{per}(\alpha))$ is some element of $\mathrm{H}^5(X, \mathbb{Z})$. It must be a torsion element since $\mathrm{ind}_{\mathbf{K}}(\alpha) = \mathrm{ind}(\alpha)$ is finite as X is a finite CW-complex. So, again, it is zero. This gives $\mathrm{ind}(\alpha) = \mathrm{ind}_{\mathbf{K}}(\alpha) = \mathrm{per}(\alpha)$.

In the algebraic case, there is an analogue of the intermediary $\mathrm{ind}_{\mathbf{K}}$ that sits between per and ind.

Definition 2.25. The étale index $\mathrm{eti}(\alpha)$ of a class $\alpha \in \mathrm{Br}'(X_{\text{ét}})$ was defined in [4] as the positive generator of the rank map $a_{\text{ét}}\mathbf{K}_0^{\alpha}(X) \to \mathbb{Z}$.

Corollary 2.26. *For X a scheme and* $\alpha \in \mathrm{Br}(X_{\text{ét}})$*,*

$$\mathrm{ind}_{\mathbf{K}}(\alpha_{\mathrm{top}}) | \mathrm{eti}(\alpha).$$

Proof. This follows from 2.11. ☐

3 The prime-divisor problem for topological spaces

In this short section we prove that for a finite CW complex X and for $\alpha \in \mathrm{Br}(X)$, the primes dividing $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ agree.

Let A be a finitely-generated abelian group and r a positive integer, not necessarily prime. We say $a \in A$ is r-primary if $r^m a = 0$ for some positive integer m. We say A is r-primary if all the elements of A are r-primary. For a given positive integer r, the class of r-primary finitely-generated abelian groups is a Serre class; it is closed under the taking of subobjects and the formation of quotients and extensions. We denote this Serre class by $\mathcal{C}_r$. There is a mod-$\mathcal{C}_r$ Hurewicz theorem, [49], which implies that if X is a simply-connected CW-complex and if every group $\pi_n(X)$ with $n > 1$ is r-primary, then so too is every homology group $\mathrm{H}_n(X, \mathbb{Z})$ with $n > 0$. In particular, using the universal coefficients theorem, we know that $\mathrm{H}^n(K(\mathbb{Z}/r, 2), \mathbb{Z})$ is r-primary for all $n > 0$.

Since $K(\mathbb{Z}/r, 2)$ is the universal space for r-torsion classes in $\mathrm{H}^3(\cdot, \mathbb{Z})$, we can exploit the r-primary nature of the cohomology here to give a general statement concerning the differentials in the Atiyah-Hirzebruch spectral sequence of proposition 2.21.

Theorem 3.1. *If X is a CW-complex and if $\alpha \in \mathrm{Br}(X)$, then $\mathrm{ind}_{\mathbf{K}}(\alpha)$ and $\mathrm{per}(\alpha)$ have the same prime divisors.*

Proof. Since $\mathrm{per}(\alpha) | \mathrm{ind}_{\mathbf{K}}(\alpha)$, it suffices to show that any prime divisor of $\mathrm{ind}_{\mathbf{K}}(\alpha)$ is also a prime divisor of $\mathrm{per}(\alpha)$.

That $\mathrm{ind}_{\mathbf{K}}(\alpha)$ is finite follows from the fact that α is in the Brauer group. Let $\mathrm{per}(\alpha) = r$ so that α is the image under the unreduced Bockstein map of a class $\beta \in \mathrm{H}^2(X, \mathbb{Z}/r)$. In particular, there is a map $\phi : X \to K(\mathbb{Z}/r, 2)$ such that $\alpha \in \mathrm{H}^3(X, \mathbb{Z})$ is the pull-back of $\beta \in \mathrm{H}^3(K(\mathbb{Z}/r, 2), \mathbb{Z})$ under ϕ.

We recall from proposition 2.21 that $\mathrm{ind}_{\mathbf{K}}(\alpha)$ is a generator for the ideal of permanent cycles in $\mathrm{E}_2^{0,0}$. The map ϕ induces a map of Atiyah-Hirzebruch spectral sequences, which we denote by Φ.

$$\begin{tikzcd} {}_I\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathbb{Z}(q/2)) \arrow[r, Rightarrow] & \mathrm{KU}^{p+q}(X)_\alpha \\ {}_{II}\mathrm{E}_2^{p,q} = \mathrm{H}^p(K(\mathbb{Z}/r, 2), \mathbb{Z}(q/2)) \arrow[r, Rightarrow] \arrow[u, "\Phi"] & \mathrm{KU}^{p+q}(K(\mathbb{Z}/r, 2))_\beta. \end{tikzcd}$$

We observe that $\Phi : {}_{II}\mathrm{E}_2^{0,0} = \mathbb{Z} \to {}_I\mathrm{E}_2^{0,0} = \mathbb{Z}$ is simply the isomorphism on H^0 induced by ϕ. Subsequently, we find ${}_I\mathrm{E}_n^{0,0} \subset {}_I\mathrm{E}_2^{0,0}$ and similarly ${}_{II}\mathrm{E}_n^{0,0} \subset {}_{II}\mathrm{E}_2^{0,0}$; for dimensional reasons there are no differentials whose target is $\mathrm{E}_n^{0,0}$ in either spectral sequence.

A routine homological-algebra argument suffices to show that $\Phi : {}_{II}\mathrm{E}_n^{0,0} \to {}_I\mathrm{E}_n^{0,0}$ is injective for all n, we are therefore justified in writing

$$ {}_{II}\mathrm{E}_n^{0,0} = m_n'\mathbb{Z} \subset {}_I\mathrm{E}_n^{0,0} = m_n\mathbb{Z} \subset {}_I\mathrm{E}_2^{0,0} = \mathbb{Z} \quad (1)$$

where m_n and m_n' are chosen to be nonnegative.

We claim that the prime factors of the integer m'_n are prime factors of r. By definition we have exact sequences

$$0 \longrightarrow {}_{II}\mathrm{E}_{n+1}^{0,0} = m'_{n+1}\mathbb{Z} \longrightarrow {}_{II}\mathrm{E}_n^{0,0} = m'_n\mathbb{Z} \xrightarrow{d_n} {}_{II}\mathrm{E}_n^{n,-n+1}$$

Here ${}_{II}\mathrm{E}_n^{n,-n+1}$ is a subquotient of ${}_{II}\mathrm{E}_2^{n,-n+1} = \mathrm{H}^n(K(\mathbb{Z}/r,2),\mathbb{Z}(\frac{-n+1}{2}))$, and since the latter is r-primary, so is the former. In particular, the image of d_n in the above sequence is r-primary, but this image is isomorphic to the group $\mathbb{Z}/(m'_{n+1}/m'_n)$, and consequently m'_{n+1}/m'_n is a positive integer whose prime factors all divide r. The claim follows by induction.

From equation (1), we conclude that $m_n | m'_n$ and so the prime factors of m_n number among the prime factors of r. Finally, we observe that since $\mathrm{ind}_{\mathbf{K}}(\alpha)$ is finite, the we have a nonzero group on the E_∞-page, that is

$$0 \subsetneq {}_I\mathrm{E}_\infty^{0,0} \cong \mathrm{ind}_{\mathbf{K}}(\alpha)\mathbb{Z} \subset {}_I\mathrm{E}_2^{0,0} = \mathbb{Z}$$

It follows that $\mathrm{ind}_{\mathbf{K}}(\alpha) = m_N$ for some sufficiently large N, and so we conclude that the prime numbers dividing $\mathrm{ind}_{\mathbf{K}}(\alpha)$ also divide $r = \mathrm{per}(\alpha)$, as required. □

Corollary 3.2. *If X is a finite CW-complex, and if $\alpha \in \mathrm{Br}(X)$, then the primes dividing $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ agree.*

Proof. This follows immediately from the theorem, since $\mathrm{ind}_{\mathbf{K}}(\alpha) = \mathrm{ind}(\alpha)$. □

4 The period-index problem for topological spaces

The following theorem provides our main upper-bound.

Theorem 4.1. *Suppose that X is a d-dimensional CW-complex, and let*

$$\alpha \in \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tors}}.$$

Then, $\mathrm{ind}_{\mathbf{K}}(\alpha)$ is finite, and

$$\mathrm{ind}_{\mathbf{K}}(\alpha) | \prod_{j=1}^{d-1} e_j^\alpha$$

where e_j^α is the exponent of $\pi_j^s(B\mathbb{Z}/\mathrm{per}(\alpha))$, where π_j^s denotes stable homotopy.

Proof. Let β be a lift of α to $\mathrm{H}^2(X,\mathbb{Z}/r)$, where $r = \mathrm{per}(\alpha)$. Then, by proposition 2.19, there is a map of presheaves of spectra $\mathrm{S}[\mathbb{Z}/r](X)_\beta \to \mathrm{KU}(X)_\alpha$. Consider the spectral sequence of corollary 2.20

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \pi_q\mathrm{S}[\mathbb{Z}/r]) \Rightarrow \pi_{q-p}\mathrm{S}[\mathbb{Z}/m]^\wedge(X)_\beta,$$

where β is a lift of α. The differentials d_k are of degree $(k, k-1)$. The groups $\pi_q\mathrm{S}[\mathbb{Z}/r]$ are $\pi_q^s \oplus \pi_q B\mathbb{Z}/r$ by lemma 2.16. Write ℓ_q^α for the exponent of $\pi_q\mathrm{S}[\mathbb{Z}/r]$.

Because of the cohomological dimension, the last possible non-zero differential coming from $\mathrm{E}^{0,0}$ is d_d. Thus, we are only concerned with $\pi_q \mathrm{S}[\mathbb{Z}/r]$ up to $q = d-1$. But, the differential d_k^β leaving $\mathrm{E}_r^{0,0}$ lands in a group of exponent at most ℓ_{k-1}^α. Therefore,

$$\prod_{j=1}^{d-1} \ell_j^\alpha$$

is a permanent cycle in the twisted spectral sequence for $\mathrm{S}[\mathbb{Z}/m]^\wedge(X)_\beta$. Since there is a morphism of the spectral sequence for $\mathrm{S}[\mathbb{Z}/m]^\wedge(X)_\beta$ to $\mathrm{KU}(X)_\alpha$, it follows that the product is a permanent cycle in the twisted spectral sequence for $\mathrm{KU}(X)_\alpha$. By theorem 3.1, $\mathrm{ind}_{\mathbf{K}}(\alpha)$ divides the r-primary part of

$$\prod_{j=1}^{d-1} \ell_j^\alpha.$$

By the second part of lemma 2.16, the r-primary part of ℓ_j^α is the exponent of $\pi_j^s(B\mathbb{Z}/r)$. □

Corollary 4.2. *If X is a finite CW-complex of dimension d, then theorem 4.1 and corollary* **??** *hold with* $\mathrm{ind}(\alpha)$ *in place of* $\mathrm{ind}_{\mathbf{K}}(\alpha)$.

Proof. This follows from lemma 2.22. □

Corollary 4.3. *Let X be a Stein space having the homotopy type of a finite CW-complex. Then theorem 4.1 and corollary* **??** *hold the analytic index in place of* $\mathrm{ind}_{\mathbf{K}}(\alpha)$.

Proof. This follows from the Oka principle [24]. □

Now, we analyze the integers e_j^α in a certain range, when $\mathrm{per}(\alpha)$ is a prime-power ℓ^n.

Proposition 4.4. *Let $0 < k < 2\ell - 3$. Then, the ℓ-primary component $\pi_k^s(\ell)$ of π_k^s is zero. And,*

$$\pi_{2l-3}^s(\ell) = \mathbb{Z}/\ell.$$

Proof. A proof of this can be found in [44, Theorems 1.1.13-14]. □

Proposition 4.5. *For $0 < k < 2\ell - 2$, the stable homotopy group $\pi_k^s(B\mathbb{Z}/\ell^n)$ is isomorphic to $\mathbb{Z}/\ell^n$ for k odd and zero for k even.*

Proof. This is [4, Proposition 4.2]. □

Theorem 4.6. *Let X be a d-dimensional CW-complex, let ℓ be a prime such that $2\ell > d+1$, and let* $\alpha \in \mathrm{Br}'(X) = \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ *satisfy* $\mathrm{per}(\alpha) = \ell^k$*; then*

$$\mathrm{ind}_{\mathbf{K}}(\alpha) | \operatorname{per}(\alpha)^{[\frac{d}{2}]}.$$

Proof. This follows immediately from theorem 4.1 and the previous two propositions. □

Corollary 4.7. *Let X be a finite CW-complex of dimension d, let ℓ be a prime such that $2\ell > d+1$, and suppose $\alpha \in \mathrm{Br}(X) = \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$ satisfies $\mathrm{per}(\alpha) = \ell^k$; then,*

$$\mathrm{ind}(\alpha) | \, \mathrm{per}(\alpha)^{[\frac{d}{2}]}.$$

Corollary 4.8. *Let X be a d-dimensional Stein space having the homotopy type of a finite CW-complex, and let ℓ be a prime such that $2\ell > d+1$. If $\alpha \in \mathrm{Br}(X)$ satisfies $\mathrm{per}(\alpha) = \ell^k$; then,*

$$\mathrm{ind}(\alpha_{\mathrm{an}}) | \, \mathrm{per}(\alpha)^{[\frac{d}{2}]},$$

where $\mathrm{ind}(\alpha_{\mathrm{an}})$ is the ℓ-part of the greatest common divisor of the degrees of all analytic Azumaya algebras in the class α.

Remark 4.9. The Atiyah-Hirzebruch spectral sequence says that for X a finite CW-complex of dimension at most 4, per = ind. Therefore, the theorem is not sharp, in general.

5 Lower Bounds on the Index

We first establish a number-theoretic result that will apply in the study of the cohomology of PU_n. We then consider the problem of finding a degree-n representative for a class in $\mathrm{Br}'(X) = \mathrm{H}^3(X, \mathbb{Z})_{\mathrm{tors}}$, which is the same as a factorization of $X \to K(\mathbb{Z}, 3)$ as $X \to BPU_n \to K(\mathbb{Z}, 3)$. We obtain a family of obstructions to such a factorization that can most easily be computed after application of the reduced loopspace functor, $\Omega(\cdot)$, and we then use this family to furnish examples.

5.1 Calculations in elementary number-theory

Definition 5.1. We define an integer-valued function $m : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$m(a, s) = \gcd \left\{ \binom{a}{i} \right\}_{i=1}^{s}$$

where the binomial coefficient $\binom{a}{i} = 0$ if $i > a$.

We observe that $m(a, s)$ is a decreasing function of s, in that $m(a, s+1) | m(a, s)$ for all s, it begins with $m(a, 1) = a$ and stabilizes at $m(a, a) = 1$.

In the following lemma and subsequently, the notation $[x]$ will be used to denote the integral part of the real number x.

Lemma 5.2. *Let p be a prime number, and let n, s be positive integers. Write $c = \max\{n - [\log_p s], 0\}$, then $m(p^n, s) = p^c$.*

Proof. It suffices to determine the power of p dividing $m(p^n, s)$.

A theorem of Kummer [36] says that the exponent of p dividing $\binom{a+b}{b}$ is equal to the number of carries that arise in the addition of a and b in base-p arithmetic. From this, we deduce that the exponent of p dividing $\binom{p^n}{k}$ is at least $n - [\log_p k]$, although it may be more. For $k = p^j$, where $j \leq n$, Kummer's result says this inequality is in fact an equality. Since the exponent of p dividing $m(p^n, s)$ is the infimum of the exponents of p dividing $\{\binom{p^n}{k}\}_{k=1}^s$, the stated result follows. □

Lemma 5.3. *Let p be a prime number, and let a be a positive integer relatively prime to p. Let s, n be positive integers. Then the power of p dividing $m(ap^n, s)$ coincides with the power of p dividing $m(p^n, s)$.*

Proof. It suffices to prove this for $s \leq p^n$, because $m(p^n, p^n) = 1$, and $m(ap^n, s+1) | m(ap^n, s)$ for all s, so that if the result holds for $s = p^n$, it will hold trivially for $s > p^n$.

Assume therefore that $s \leq p^n$. We claim that the power of p dividing $\binom{ap^n}{k}$ coincides with that dividing $\binom{p^n}{k}$ for all $k \leq s$. If $0 \leq r < k$, then $ap^n - r$ is not divisible by p^{n+1}, for otherwise we should have r divisible by p^n, which can happen within the constraint $r < p^n$ only if $r = 0$, in which case $ap^n - r = ap^n$ which is not divisible by p^{n+1} by assumption.

In the binomial expansion

$$\binom{ap^n}{k} = \frac{(ap^n)(ap^n - 1) \dots (ap^n - k + 1)}{k!}$$

the power of p dividing a term $ap^n - r$ on the top row is no greater than p^n, and therefore coincides with the power of p dividing r, and also with the power of p dividing $p^n - r$. Comparing, term-by-term, we see that the power of p dividing $\binom{ap^n}{k}$ coincides with that dividing

$$\binom{p^n}{k} = \frac{p^n(p^n - 1) \dots (p^n - k + 1)}{k!}$$

It follows that the power of p dividing $m(ap^n, s)$ coincides with that dividing $m(p^n, s)$, as claimed. □

Corollary 5.4. *For a positive integer a, with prime factorization $a = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$, and for a given positive integer s, we have*

$$m(a, s) = \prod_{i=1}^{r} p_i^{\max\{n_i - [\log_{p_i} s], 0\}}$$

Proof. Once one observes that the only prime factors $m(a, s)$ may have are the primes dividing $m(a, 1) = a$, this result follows immediately from the previous two lemmas. □

Definition 5.5. Let b, s be positive integers. Define $n(b, s)$ as follows. Write $b = \prod_{i=1}^{r} p_i^{n_i}$, where the p_i are unique primes and the n_i are positive integers. Define

$$n(b, s) = \prod_{i=1}^{r} p_i^{n_i + [\log_{p_i} s]}$$

Corollary 5.6. *Let b, a and s be positive integers. If $b \mid m(a, s)$, then $n(b, s) \mid a$.*

Proof. Write $b = \prod_{i=1}^{r} p_i^{n_i}$ as in the definition of $n(b, s)$. Write $a = a' \prod_{i=1}^{r} p_i^{n'_i}$, where a' is relatively prime to b and where the n'_i are not necessarily positive. Then $p_i^{n_i} \mid p_i^{n'_i - \max\{0, [\log_{p_i} s]\}}$, so that in particular $n_i + [\log_{p_i} s] < n'_i$, from which the claim follows. □

Note that for fixed b, $\lim_{s \to \infty} n(b, s) = \infty$.

5.2 Obstructions arising from the cohomology suspension

In this section we obtain obstructions to factorizations $X \to BPU_a \to K(\mathbb{Z}, 3)$ of a-torsion classes by finding obstructions that survive an application of the loop-space functor $\Omega(X) \to PU_a \to K(\mathbb{Z}, 2)$. Not only is the integral cohomology of PU_a slightly easier to describe than that of BPU_a, it also happens that the product structure on the cohomology of PU_a yields a number of obstructions without our having to resort to cohomology operations.

To this end, we first describe what we need of the cohomology of PU_a. Thereafter the description of lower bounds on the index in theorem 5.8 is straightforward, and it is easy to use these bounds to give examples where the index is arbitrarily large compared to the period.

For an element ζ in an abelian group H, let $\mathrm{ord}(\zeta)$ denote the order of ζ in H.

Proposition 5.7. *With integer coefficients, $\mathrm{H}^1(PU_a, \mathbb{Z}) = 0$ and $\mathrm{H}^2(PU_a, \mathbb{Z}) = \mathbb{Z}/a$. Fix a generator, $\eta \in \mathrm{H}^2(PU_a, \mathbb{Z})$. Then,*

$$\mathrm{ord}(\eta^s) = m(a, s),$$

In particular, $\mathrm{ord}(\eta^a) = 1$, so $\eta^a = 0$.

Proof. Consider the fibration

$$U_a \to PU_a \to BS^1.$$

The integral cohomology of U_a is

$$\mathrm{H}^*(U_a, \mathbb{Z}) = \Lambda_{\mathbb{Z}}(\alpha_1, \dots, \alpha_a),$$

an exterior algebra in a generators with the degree of α_i being $2i - 1$, while

$$\mathrm{H}^*(BS^1, \mathbb{Z}) = \mathbb{Z}[\theta],$$

where the degree of θ is 2. Consider the spectral sequence of the fibration

$$\mathrm{H}^p(BS^1, \mathrm{H}^q(U_a, \mathbb{Z})) \Rightarrow \mathrm{H}^{p+q}(PU_a, \mathbb{Z}).$$

Then, [9, Theorem 4.1] says that $d_{2i}(\alpha_i) = \binom{a}{i}\theta^i$. By induction and the algebra structure on the spectral sequence, it follows that the class η, which is the image of θ, has order as stated. It also follows from the spectral sequence that $\mathrm{H}^1(PU_a, \mathbb{Z}) = 0$ and $\mathrm{H}^2(PU_a, \mathbb{Z}) = (\mathbb{Z}/a)\eta$, as claimed. □

Recall that if $\alpha \in \tilde{\mathrm{H}}^n(X, A)$, then α may be represented as a based map $X \to K(A, n)$. Applying the reduced loopspace functor, $\Omega(\cdot)$, one obtains a natural transformation of functors

$$\Omega : \tilde{\mathrm{H}}^n(\cdot, A) \to \tilde{\mathrm{H}}^{n-1}(\Omega(\cdot), A)$$

This natural transformation is termed the *cohomology suspension*.

One observes from the Serre spectral sequence associated with the fibration $PU_a \to EPU_a \to BPU_a$ that $\mathrm{H}^3(BPU_a, \mathbb{Z}) = \mathbb{Z}/a$, generated by a class $\tilde{\eta}$, which may be chosen in such a way that $\Omega(\tilde{\eta}) : \Omega(BPU_a) \to \Omega(K(\mathbb{Z}, 3))$ is the distinguished generator η.

Theorem 5.8. *Let $\tilde{\alpha} \in \mathrm{H}^3(X, \mathbb{Z})$ be a cohomology class, and let $\alpha \in \mathrm{H}^2(\Omega X, \mathbb{Z})$ be the cohomology suspension of this class. If $X \to K(\mathbb{Z}, 3)$ factors through $\tilde{\eta} : BPU_a \to K(\mathbb{Z}, 3)$ then α^s is $m(a, s)$-torsion for all $s \geq 1$.*

Proof. If $\tilde{\alpha} : X \to K(\mathbb{Z}, 3)$ factors as $X \to BPU_a \to K(\mathbb{Z}, 3)$, then applying $\Omega(\cdot)$ shows that $x : \Omega(X) \to K(\mathbb{Z}, 2)$ factors through $\eta : PU_a \to K(\mathbb{Z}, 2)$. The result now follows from our previous determination of the order of η. □

Remark 5.9. For $s = 1$, this theorem indicates that α is a-torsion. For $s = a$, it means that α^a is 1-torsion, i.e. zero, so x must in particular be nilpotent.

Let β denote the *unreduced Bockstein*, any one of the natural transformations of cohomology groups that appears as the boundary map in the long exact sequence

$$\longrightarrow \tilde{\mathrm{H}}^i(X, \mathbb{Z}) \xrightarrow{\times r} \tilde{\mathrm{H}}^i(X, \mathbb{Z}) \longrightarrow \tilde{\mathrm{H}}^i(X, \mathbb{Z}/r) \xrightarrow{\beta} \tilde{\mathrm{H}}^{i+1}(X, \mathbb{Z}) \longrightarrow$$

Each unreduced Bockstein is a natural transformation of cohomology functors, and consequently may be represented by a map $\beta : K(\mathbb{Z}/r, i) \to K(\mathbb{Z}, i+1)$

Corollary 5.10. *The Bockstein $\beta : K(\mathbb{Z}/r, 2) \to K(\mathbb{Z}, 3)$ does not factor through any $BPU_a \to K(\mathbb{Z}, 3)$.*

Proof. The following argument appears for $r = 2$ in Atiyah-Segal [7, proof of Proposition 2.1.v]. The class β has additive order r. Since $K(\mathbb{Z}/r, 2)$ is simply-connected and since $\mathrm{H}_2(K(\mathbb{Z}/r, 2), \mathbb{Z})$ is torsion, it follows from the universal-coefficients theorem that $\tilde{\mathrm{H}}^i(K(\mathbb{Z}/r, 2), \mathbb{Z}) = 0$ for $i < 3$. The usual arguments [43] used to show that the cohomology suspension is an isomorphism apply

in this instance to show that the cohomology suspension $\mathrm{H}^3(K(\mathbb{Z}/r,2),\mathbb{Z}) \to \mathrm{H}^2(K(\mathbb{Z}/r,1),\mathbb{Z})$ is an isomorphism. In particular $\xi = \Omega(\beta)$ is a generator of $\mathrm{H}^2(K(\mathbb{Z}/r,1),\mathbb{Z}) = (\mathbb{Z}/r)\xi$. It is generally known that $\mathrm{H}^*(K(\mathbb{Z}/r,1),\mathbb{Z}) = \mathrm{H}^*(B\mathbb{Z}/r,\mathbb{Z})$ is $\mathbb{Z}[\xi]/(r\xi)$, see [53, Chapter 6], from which it follows that $\mathrm{ord}(\xi^n) = r$ for all $n \geq 1$. In particular ξ is not nilpotent in $\mathrm{H}^*(K(\mathbb{Z}/r,1),\mathbb{Z})$. □

Recall that a map $f : X \to Y$ is an n-equivalence if $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k < n$ and is surjective for $k = n$.

Lemma 5.11. *Let $f : X \to Y$ be an n-equivalence. Then,*

$$f^* : \mathrm{H}^k(Y,\mathbb{Z}) \to \mathrm{H}^k(X,\mathbb{Z})$$

is an isomorphism for $k < n$ and is an injection for $k = n$.

Proof. By the Whitehead theorem [51, Theorem 10.28], f induces an isomorphism on integral homology in degrees less than n and a surjection in degree n. Now, apply the universal-coefficients theorem

$$0 \to \mathrm{Ext}(\mathrm{H}_{k-1}(Y),\mathbb{Z}) \to \mathrm{H}^k(Y,\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_k(Y,\mathbb{Z}),\mathbb{Z}) \to 0.$$

□

Corollary 5.12. *Let X be a CW-complex, and let $\mathrm{sk}_{n+1} X$ denote the $n+1$-skeleton of X, so that $i : \mathrm{sk}_{n+1} X \to X$ is an n-equivalence by cellular approximation. Then,*

$$\Omega\, \mathrm{sk}_{n+1} X \to \Omega X$$

is an $n-1$-equivalence, and so $\mathrm{H}^k(\Omega X,\mathbb{Z}) \to \mathrm{H}^k(\Omega\, \mathrm{sk}_{n+1} X,\mathbb{Z})$ is an isomorphism for $k < n-1$ an injection for $k = n-1$.

Proof. Taking loops sends n-equivalences to $n-1$-equivalences. □

Example 5.13. Fix a positive integer r. Consider the Eilenberg-MacLane space $K(\mathbb{Z}/r,2)$. We know there is a class $\beta \in \mathrm{H}^3(K(\mathbb{Z}/r,2),\mathbb{Z})$ such that the cohomology suspension $\Omega(\beta)$ is a generator, ξ, for $\mathrm{H}^2(K(\mathbb{Z}/r,1),\mathbb{Z})$. We also know that $\mathrm{ord}(\xi^n) = r$ for all $n \geq 1$, as was mentioned in the proof of corollary 5.10.

If we take instead the CW-complex $f_a : X_{a,r} = \mathrm{sk}_{a+1} K(\mathbb{Z}/r,2) \to K(\mathbb{Z}/r,2)$, with $a \geq 3$, which may be assumed to be a finite CW-complex, then there are isomorphisms arising from the inclusion of the skeleton

$$f_a^* : \mathrm{H}^i(K(\mathbb{Z}/r,2),\mathbb{Z}) \overset{\simeq}{\to} \mathrm{H}^i(X_{a,r},\mathbb{Z}) \qquad \text{for } i < a$$
$$\Omega(f_a)^* : \mathrm{H}^{i-1}(K(\mathbb{Z}/r,1),\mathbb{Z}) \overset{\simeq}{\to} \mathrm{H}^{i-1}(\Omega(X_{a,r}),\mathbb{Z}) \qquad \text{for } i < a$$

Denote the class $f_a^*(\beta)$ by γ_a. Since $a \geq 3$, γ_a has order r. Denote the cohomology suspension $\Omega(\gamma_a)$ by α. The cohomology suspension being natural, we have $\alpha = \Omega(\gamma_a) = f^*(\Omega(\beta)) = f^*(\xi)$. In particular, α^j has order r provided it lies in the range where f_a^* is an inclusion, i.e. provided it lies in $\mathrm{H}^i(\Omega(X_{a,r},\mathbb{Z})$

with $i \leq a-1$, which is to say $j \leq \frac{a-1}{2}$. If therefore $\gamma_a : X_{a,r} \to K(\mathbb{Z}, 3)$ is to factor $X_{a,r} \to BPU_N \to K(\mathbb{Z}, 3)$, we must have $r \mid m(N, [\frac{a-1}{2}])$, from which it follows that $n(r, [\frac{a-1}{2}]) \mid N$ by corollary 5.6. In particular

$$\mathrm{per}(\gamma_a) = r \qquad \text{but} \qquad n\left(r, \left[\frac{a-1}{2}\right]\right) \mid \mathrm{ind}(\gamma_a)$$

Letting $a \to \infty$, we obtain in this way for a given r, a sequence of spaces $X_{a,r}$ having in each case a class $\gamma_a \in \mathrm{H}^3(X_{a,r}, \mathbb{Z})$ such that $\mathrm{per}(\gamma_a) = r$ but $\mathrm{ind}(\gamma_a) \to \infty$ as $a \to \infty$.

Example 5.14. Using the example and theorem 4.1, one finds that for the class γ_5 just considered on $\mathrm{sk}_6\, K(\mathbb{Z}/2, 2)$,

$$2^2 | \,\mathrm{ind}(\gamma_3) | 2^6.$$

In this range, π_i^s for $i = 1, \ldots, 5$, the stable homotopy of $B\mathbb{Z}/2$ is just the 2-primary part of the stable homotopy of spheres, except for $\pi_4^s B\mathbb{Z}/2 = \mathbb{Z}/2$.

5.3 Remarks on the descent spectral sequence for $K(\mathbb{Z}/r, 2)$

It was shown in corollary 5.10 that $\mathrm{ind}(\beta) = \infty$ for the Bockstein $\beta \in \mathrm{Br}'(K(\mathbb{Z}/r, 2))$. Now, suppose that $\mathrm{ind}_{\mathbf{K}}(\beta)$ were finite. Then, $\mathrm{ind}_{\mathbf{K}}(\gamma)$ would be bounded by $\mathrm{ind}_{\mathbf{K}}(\beta)$ for every r-torsion class γ on every space X, whereas above there are examples of finite CW-complexes and classes γ with $\mathrm{per}(\gamma) = r$ and $\mathrm{ind}(\gamma) = \mathrm{ind}_{\mathbf{K}}(\gamma)$ arbitrarily large. It follows that $\mathrm{ind}_{\mathbf{K}}(\beta)$ is infinite.

Let c_k be the non-negative generator of $\mathrm{E}_k^{0,0} \subseteq \mathrm{H}^0(K(\mathbb{Z}/r, 2), \mathbb{Z})$ in the descent spectral sequence for $\mathrm{KU}(K(\mathbb{Z}/r, 2))_\beta$. The elements $d_k(c_k)$ are obstructions to $\mathrm{ind}(\beta) = c_k$. Note that $c_2 = 1$ and $d_2^\beta(1) = \beta$.

Proposition 5.15. *The obstructions to the finiteness of* $\mathrm{ind}_{\mathbf{K}}(\beta)$ *in the descent spectral sequence on* $K(\mathbb{Z}/r, 2)$ *are all of finite order in* $\mathrm{E}_k^{k,1-k}$*, and infinitely many of them are non-zero.*

Proof. In fact, the obstructions all lie in subquotients of $\mathrm{H}^p(K(\mathbb{Z}/r, 2), \mathbb{Z}(q/2))$ with $p > 0$, which are finitely generated groups, and were observed in section 3 to be r-primary, hence torsion, and consequently finite.

If the obstructions were to vanish for sufficiently large k, then $\mathrm{ind}_{\mathbf{K}}(\beta)$ would be finite, a contradiction. $\square$

Fix a prime ℓ. The first potentially non-zero obstruction in the twisted spectral sequence for $K(\mathbb{Z}/\ell, 2)$ is $d_3^\beta(1)$. This is in fact non-zero. Namely, $d_3^\beta(1) = \beta \in \mathrm{H}^3(K(\mathbb{Z}/\ell, 2), \mathbb{Z})$. It is important to know for computations the next non-zero obstruction in the twisted spectral sequence of $K(\mathbb{Z}/\ell, 2)$. The following proposition gives a partial answer for all primes ℓ, and, together with remark 5.14, this shows that $d_5^\beta(2)$ is non-zero for $K(\mathbb{Z}/2, 2)$.

Proposition 5.16. *The next non-zero obstruction in the twisted spectral sequence for* $K(\mathbb{Z}/\ell, 2)$ *is one of* $d_5^\beta(\ell \cdot 1), \ldots, d_{2\ell+1}^\beta(\ell \cdot 1)$.

Proof. One knows from example 5.13 that the index of the class $\gamma_{2\ell+1}$ on

$$\mathrm{sk}_{2\ell+2}\, K(\mathbb{Z}/\ell, 2)$$

is at least ℓ^2. If the differentials $d_5^\beta(\ell \cdot 1), \dots, d_{2\ell+1}^\beta(\ell \cdot 1)$ were all zero, then, by lemma 2.22, the descent spectral sequence would give $\mathrm{ind}(\gamma_{2\ell+1}) = \ell$, a contradiction. □

Corollary 5.17. *In the twisted spectral sequence for* $K(\mathbb{Z}/2, 2)$*, the obstruction* $d_5^\beta(2)$ *is non-zero.*

Proof. This is a special case of the proposition. □

Corollary 5.18. *If* X *is a finite CW-complex of dimension at most* 6*, and if* $\alpha \in \mathrm{Br}(X_{\mathrm{top}})$ *has* $\mathrm{per}(\alpha) = 2$*, then* $\mathrm{ind}(\alpha)|8$*.*

Proof. It follows from [12, Théorème 4] and an easy computation in the Serre spectral sequence for $K(\mathbb{Z}/2, 1) \to * \to K(\mathbb{Z}/2, 2)$ that $\mathrm{H}^5(K(\mathbb{Z}/2, 2), \mathbb{Z}) = \mathbb{Z}/4$. □

In this case our knowledge of the cohomology of $K(\mathbb{Z}/2, 2)$ gives us better bounds than are obtained via theorem 4.1. In general, however, it is not clear to us whether studying the cohomology of $K(\mathbb{Z}/r, 2)$ should give tighter bounds than 4.1. We will return to this question in a future work.

References

[1] M. Ando, A. J. Blumberg, D. J. Gepner, M. J. Hopkins, and C. Rezk, *Units of ring spectra and Thom spectra*, ArXiv e-prints (2008), 0810.4535. 2.4

[2] Matthew Ando, Andrew J. Blumberg, and David Gepner, *Twists of K-theory and TMF*, Superstrings, geometry, topology, and C^*-algebras, Proc. Sympos. Pure Math., vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 27–63. MR2681757 2.4, 2.13, 2.19

[3] B. Antieau, *Cech approximation to the Brown-Gersten spectral sequence*, ArXiv e-prints (2009), 0912.3786, to appear in Homology, Homotopy Appl. 2.4

[4] Benjamin Antieau, *Cohomological obstruction theory for Brauer classes and the period-index problem*, J. K-Theory (2010), Available on CJO 13 Dec 2010 `doi:10.1017/is010011030jkt136`. 1, 1, 1, 2, 2.3, 2.11, 2.25, 4.5

[5] M. Artin, *Brauer-Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 194–210. MR657430 1

[6] M. Artin and D. Mumford, *Some elementary examples of unirational varieties which are not rational*, Proc. London Math. Soc. (3) **25** (1972), 75–95. MR0321934 1.4

[7] Michael Atiyah and Graeme Segal, *Twisted K-theory*, Ukr. Mat. Visn. **1** (2004), no. 3, 287–330. MR2172633 1, 2, 2.1, 2.2, 2.3, 2.8, 2.9, 5.10

[8] ———, *Twisted K-theory and cohomology*, Inspired by S. S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 5–43. MR2307274 1, 2.1, 2.4

[9] Paul F. Baum and William Browder, *The cohomology of quotients of classical groups*, Topology **3** (1965), 305–336. MR0189063 1, 5.7

[10] Karim Johannes Becher and Detlev W. Hoffmann, *Symbol lengths in Milnor K-theory*, Homology Homotopy Appl. **6** (2004), no. 1, 17–31 (electronic). MR2061565 1

[11] V. G. Berkovič, *The Brauer group of abelian varieties*, Funkcional Anal. i Priložen. **6** (1972), no. 3, 10–15. MR0308134 1.2

[12] H. Cartan, *Détermination des algèbres* $\mathrm{H}_*(\pi, n; \mathbb{Z})$, Séminaire H. Cartan **7** (1954/1955), no. 1, 11–01–11–24. 5.18

[13] J.-L. Colliot-Thélène, *Die brauersche gruppe ; ihre verallgemeinerungen und anwendungen in der arithmetischen geometrie*, http://www.math.u-psud.fr/~colliot/, 2001. 1

[14] ———, *Exposant et indice d'algèbres simples centrales non ramifiées*, Enseign. Math. (2) **48** (2002), no. 1-2, 127–146, With an appendix by Ofer Gabber. MR1923420 1

[15] Andrei Căldăraru, *Derived categories of twisted sheaves on Calabi-Yau manifolds*, Ph.D. thesis, Cornell University, May 2000, http://www.math.wisc.edu/~andreic/. 2.2, 2.5, 2.6, 2.11

[16] A. J. de Jong, *A result of Gabber*, unpublished preprint, www.math.columbia.edu/~dejong/papers/2-gabber.pdf, 2003. 1.2

[17] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94. MR2060023 1, 1.4

[18] P. Donovan and M. Karoubi, *Graded Brauer groups and K-theory with local coefficients*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 5–25. MR0282363 1, 2

[19] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), no. 1, 9–51. MR2034012 2.11

[20] Daniel Dugger and Daniel C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$-realizations*, Math. Z. **246** (2004), no. 4, 667–689. MR2045835 2.2

[21] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. MR1844577 1.2

[22] G. Elencwajg and M. S. Narasimhan, *Projective bundles on a complex torus*, J. Reine Angew. Math. **340** (1983), 1–5. MR691957 1.2

[23] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR1417719 2.4, 2.17

[24] Hans Grauert and Reinhold Remmert, *Theory of Stein spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 236, Springer-Verlag, Berlin, 1979, Translated from the German by Alan Huckleberry. MR580152 1, 4.3

[25] Alexander Grothendieck, *Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 46–66. MR0244269 1, 1.1, 1.2, 1.3

[26] ______, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 67–87. MR0244270 1.3

[27] Raymond T. Hoobler, *Brauer groups of abelian schemes*, Ann. Sci. École Norm. Sup. (4) **5** (1972), 45–70. MR0311664 1.2

[28] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208. MR1695653 2.4

[29] Daniel Huybrechts and Stefan Schröer, *The Brauer group of analytic $K3$ surfaces*, Int. Math. Res. Not. (2003), no. 50, 2687–2698. MR2017247 1.2

[30] Birger Iversen, *Brauer group of a linear algebraic group*, J. Algebra **42** (1976), no. 2, 295–301. MR0439855 1.2

[31] J. F. Jardine, *Presheaves of symmetric spectra*, J. Pure Appl. Algebra **150** (2000), no. 2, 137–154. MR1765868 2.4

[32] Daniel S. Kahn and Stewart B. Priddy, *On the transfer in the homology of symmetric groups*, Math. Proc. Cambridge Philos. Soc. **83** (1978), no. 1, 91–101. MR0464229 2.16

[33] ______, *The transfer and stable homotopy theory*, Math. Proc. Cambridge Philos. Soc. **83** (1978), no. 1, 103–111. MR0464230 2.16

[34] M. Karoubi, *Twisted bundles and twisted K-theory*, ArXiv e-prints (2010), 1012.2512. 2.2, 2.5, 2.6, 2.8

[35] Andrew Kresch, *Hodge-theoretic obstruction to the existence of quaternion algebras*, Bull. London Math. Soc. **35** (2003), no. 1, 109–116. MR1934439 1.4

[36] E. E. Kummer, *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen.*, Journal für die reine und angewandte Mathematik **1852** (1852), no. 44, 93–146. 5.1

[37] M. Lieblich, *Period and index in the Brauer group of an arithmetic surface (with an appendix by Daniel Krashen)*, ArXiv Mathematics e-prints (2007), arXiv:math/0702240. 1

[38] ———, *The period-index problem for fields of transcendence degree 2*, ArXiv e-prints (2009), 0909.4345. 1.4

[39] Max Lieblich, *Twisted sheaves and the period-index problem*, Compos. Math. **144** (2008), no. 1, 1–31. MR2388554 1, 1.4, 2.11

[40] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR2522659 2.4, 2.4

[41] ———, *Derived algebraic geometry VII: spectral schemes*, `http://www.math.harvard.edu/~lurie/`, 2011. 2.4

[42] ———, *Higher algebra*, `http://www.math.harvard.edu/~lurie/`, 2011. 2.4

[43] Robert E. Mosher and Martin C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row Publishers, New York, 1968. MR0226634 5.10

[44] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986. MR860042 4.4

[45] Jonathan Rosenberg, *Continuous-trace algebras from the bundle theoretic point of view*, J. Austral. Math. Soc. Ser. A **47** (1989), no. 3, 368–381. MR1018964 1

[46] David J. Saltman, *Division algebras over p-adic curves*, J. Ramanujan Math. Soc. (1997), no. 1, 25–47. MR1462850 1.4

[47] Stefan Schröer, *Topological methods for complex-analytic Brauer groups*, Topology **44** (2005), no. 5, 875–894. MR2153976 1.2

[48] Graeme Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR0353298 2.2, 2.9, 2.17

[49] Jean-Pierre Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258–294. MR0059548 3

[50] Richard G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277. MR0143225 2.8

[51] Robert M. Switzer, *Algebraic topology—homotopy and homology*, Springer-Verlag, New York, 1975, Die Grundlehren der mathematischen Wissenschaften, Band 212. MR0385836 5.11

[52] Robert W. Thomason, *First quadrant spectral sequences in algebraic K-theory via homotopy colimits*, Comm. Algebra **10** (1982), no. 15, 1589–1668. MR668580 2.16

[53] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 5.10

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